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1993 J. Phys. A: Math. Gen. 26 6973

(http://iopscience.iop.org/0305-4470/26/23/036)

J. Phys. A: Math. Gen. 26 (1993) 6973–6979. Printed in the UK

A new non-standard quantum supergroup

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Received 24 March 1993

Abstract. Non-standard quantum groups are very similar to quantum supergroups due to nilpotency of their elements. We introduce a non-standard quantum supergroup in which nilpotency of elements can be removed.

1. Introduction

In a recent paper [1], we introduced a multiparametric generalization of the non-standard R-matrix of $sl(n)$ and then applying the method of Faddeev–Reshetikhin–Tekhtajan (FRT) [2] to this R-matrix, we constructed the quantum group associated with it, which was denoted by $X_q(sl(n))$ in [1]. The interesting property of this quantum group was the appearance of nilpotency in its structure, which was a sign of some super structure. In this article, we apply the FRT method to the non-standard solution of the graded Yang–Baxter equation, namely the exotic solution of the GYBE corresponding to the superalgebra $sl(n|m)$ and construct the superalgebra associated with this R-matrix, which we call $X_q(sl(n|m))$. The main new features of this quantum supergroup are:

(i) Even elements may become nilpotent.

(ii) Nilpotency of odd elements may be removed.

The structure of this article is as follows: in section 2 we introduce the non-standard form of the $sl(n|m)$ R-matrix. In section 3 we apply the method FRT to this R-matrix and construct the generalization of the universal enveloping algebra of $sl(n|m)$, which we call $X_q(sl(n|m))$. $U_q(sl(n|m))$ and $U_q(sl(n+m))$ are special cases of $X_q(sl(n|m))$, and finally in section 4 we recapitulate the results (of others [3,4] and ours) in a simple example, corresponding to a (4×4) R-matrix.

2. The R-matrix

Consider a Z_2-graded vector space V with dimension N, spanned by a basis $\{e_i\}$, $i=1,\ldots,N$. $\pi(e_i)\equiv\pi_i$ is the Z_2-grade of e_i then we have:

(a) For any matrix A, the Z_2-grade of any element A_{ij} is defined as $\pi_i+\pi_j$

$$\pi_{ij}\equiv\pi_i+\pi_j.$$

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(b) The tensor product of two Z_2-graded matrices is

$$(A \otimes B)_{ij,kl} = (-1)^{\pi_k \pi_{jl}} A_{ik} B_{jl} \tag{1}$$

and the graded permutation matrix is

$$P = \sum_{i \neq j} (-1)^{\pi_i \pi_j} e_{ij} \otimes e_{ji}. \tag{2}$$

Consider the following generalization of the $sl(n|m)$ R-matrix

$$\hat{R} = \sum_{i \neq j}^{N=n+m} e_{jj} \otimes e_{ii} + \sum_{i}^{N=n+m} q_i^{\epsilon_i} e_{ii} \otimes e_{ii} + (q - q^{-1}) \sum_{i \neq j}^{N=n+m} (-1)^{\pi_i \pi_j} e_{ji} \otimes e_{ij} \tag{3}$$

where

$$\pi_i = \begin{cases} 0 & i = 1, \ldots, n \\ 1 & i = n+1, \ldots, n+m-1, \ \epsilon_i = (-1)^{\pi_i}, \ (e_{ij})_{kl} = \delta_{ik}\delta_{jl} \end{cases}$$

and each q_i can independently be equal to q or $-q^{-1}$. When a parameter q_i is $-q^{-1}$, we call it a 'twisted' parameter. The standard R-matrix of $sl(n|m)$ is obtained by setting all q_i's to q.

This R-matrix satisfies the GYBE

$$\hat{R}_{12} \hat{R}_{13} \hat{R}_{23} = \hat{R}_{23} \hat{R}_{13} \hat{R}_{12}. \tag{4}$$

In (4) one must take into account the graded nature of tensor products. The corresponding braid matrix ($B = \hat{P}\hat{R}$, where $\hat{P}$ is the graded permutation matrix) satisfies the quadratic equation

$$B^2 = (q - q^{-1})B + 1.$$

3. The structure of $X_q(sl(n|m))$

In order to obtain the quantum supergroup associated with the $\hat{R}$-matrix (3), we should solve the basic equations of FRT

$$\hat{R} L_2^{\pm} L_1^{\pm} = L_1^{\pm} L_2^{\pm} \hat{R} \qquad \hat{R} L_2^{+} L_1^{-} = L_1^{-} L_2^{+} \hat{R}. \tag{5}$$

We take the following ansatz for $L^{\pm}$ matrices

$$L^{+} = \sum_{i=1} k_i e_{ii} + \sum_{i=1} (q - q^{-1}) \left(q_i^{\epsilon_i} / q_{i+1}^{\epsilon_i+1} \right)^{1/4} (-1)^{\pi_{i,i+1}/2} (k_i k_{i+1})^{1/2} X_i^{+} e_{i,i+1}$$
$$+ \sum_{i<j-1} (q - q^{-1}) \left(q_i^{\epsilon_i} / q_j^{\epsilon_j} \right)^{1/4} (-1)^{\pi_{ij}/2} (k_i k_j)^{1/2} E_{ij}^{+} e_{ij} \tag{6}$$

$$L^{-} = \sum_{i=1} k_i^{-1} e_{ii} - \sum_{i=1} (q - q^{-1}) \left(q_i^{\epsilon_i} / q_{i+1}^{\epsilon_{i+1}} \right)^{1/4} (-1)^{\pi_{i,i+1}/2} (k_i k_{i+1})^{-1/2} X_i^{-} e_{i+1,i}$$
$$- \sum_{i-1>j}^{N} (q - q^{-1}) \left(q_i^{\epsilon_i} / q_j^{\epsilon_j} \right)^{1/4} (-1)^{\pi_{ij}/2} (k_i k_j)^{-1/2} E_{ij}^{-} e_{ij} \tag{7}$$

solution of (5) leads to

$$k_i k_j = k_j k_i \tag{8}$$

$$(\epsilon_i q_i^{\epsilon_i} - \epsilon_{i+1} q_{i+1}^{\epsilon_{i+1}})(X_i^{\pm})^2 = 0 \tag{9}$$

$$X_i^{\pm} X_j^{\pm} = X_j^{\pm} X_i^{\pm} \qquad i \geqslant j+2 \tag{10}$$

$$\frac{k_{i+1}}{k_i} X_j^{\pm} = q_i^{\pm\epsilon_i(\delta_{ij}-\delta_{i-1,j})} q_{i+1}^{\pm\epsilon_{i+1}(\delta_{ij}-\delta_{i+1,j})} X_j^{\pm} \frac{k_{i+1}}{k_i} \tag{11}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{k_{i+1}k_i^{-1} - k_{i+1}^{-1}k_i}{q - q^{-1}} \tag{12}$$

$$[X_i^+, E_{i,i+2}^+]_{q_i^{\epsilon_i}} = 0 \qquad [X_{i+1}^+, E_{i,i+2}^+]_{q_{i+1}^{-\epsilon_{i+1}}} = 0 \qquad [X_i^+, X_{i+1}^+]_{q_{i+1}^{-\epsilon_{i+1}}} = -E_{i,i+2}^+ \tag{13}$$

where $[a, b]_q = q^{1/2}ab - (-1)^{\pi_a \pi_b} q^{-1/2} ba$.

Let us identify $k_{i+1}k_i^{-1}$ with $q^{\epsilon_i H_i}\Theta_i$ where

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \qquad a_{ij} = (\delta_{ij} - \delta_{i-1,j}) + (-1)^{\pi_{i,i+1}}(\delta_{ij} - \delta_{i+1,j}) \tag{14}$$

a_{ij} is the Cartan matrix of $sl(n|m)$, H_i's are the generators of the Cartan subalgebra and the Θ_i's are the new generators in the Cartan subgroup, to be determined shortly. From (11), (14)

$$\Theta_i X_j^{\pm} = \omega_{ij}^{\pm 1} X_j^{\pm} \Theta_i \qquad \omega_{ij} = \frac{q_i^{\epsilon_i(\delta_{ij}-\delta_{i-1,j})} q_{i+1}^{\epsilon_{i+1}(\delta_{ij}-\delta_{i+1,j})}}{q^{\epsilon_i a_{ij}}}. \tag{15}$$

Thus Θ_i can be written in the following form

$$\Theta_i = \prod_{j,k} (\omega_{ij})^{a_{jk}^{-1} H_k}. \tag{16}$$

The final form of the algebra is

$$(\epsilon_i q_i^{\epsilon_i} - \epsilon_{i+1} q_{i+1}^{\epsilon_{i+1}})(X_i^{\pm})^2 = 0 \tag{17}$$

$$[H_i, H_j] = 0 \tag{18}$$

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \tag{19}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q^{\epsilon_i H_i}\Theta_i - q^{-\epsilon_i H_i}\Theta_i^{-1}}{q - q^{-1}} \tag{20}$$

$$[X_i^+, [X_i^+, X_{i+1}^+]_{q_{i+1}^{-\epsilon_{i+1}}}]_{q_i^{\epsilon_i}} = 0 \tag{21}$$

$$[X_{i+1}^+, [X_i^+, X_{i+1}^+]_{q_{i+1}^{-\epsilon_{i+1}}}]_{q_{i+1}^{\epsilon_{i+1}}} = 0 \tag{22}$$

where (21) and (22) are Serre relations and are obtained by eliminating $E_{i,i+2}^+$ from (13). In the non-standard case, where some of q_i's are equal to $-q^{-1}$ the corresponding Serre relations become trivial identities. Indicating that a basis *à la* Poincare–Birkhoff–Witt, cannot be constructed from the Chevalley–Serre presentation.

One then has to employ the full power of the FRT method and construct the algebra in the Cartan basis. For the case of $X_q(sl(n))$ this has been done in [5].

According to the general formalism of FRT this algebra is equipped with the Hopf structure

$$\Delta X_i^{\pm} = q^{-\epsilon_i H_i/2}\Theta_i^{-1/2} \otimes X_i^{\pm} + X_i^{\pm} \otimes q_i^{\epsilon_i H_i/2}\Theta_i^{1/2} \tag{23}$$

$$\Delta H_i = 1 \otimes H_i + H_i \otimes 1 \tag{24}$$

$$\epsilon(X_i^{\pm}) = \epsilon(H_i) = 0 \tag{25}$$

$$S(X_i^{\pm}) = -(q_i^{\epsilon_i} q_{i+1}^{\epsilon_{i+1}})^{\pm 1/2} X_i^{\pm} \qquad S(H_i) = -H_i \tag{26}$$

and it is clear that

$$\Delta(\Theta_i) = \Theta_i \otimes \Theta_i \qquad \epsilon(\Theta_i) = 1 \qquad S(\Theta_i) = \Theta_i^{-1}. \tag{27}$$

Note the following special cases:

(a) If $q_i = q$ and $\pi_i = 0$ then $\omega_{ij} = 1$ and the Θ_i's can be identified with unity. The relations (17)–(26) will become the usual relations of $U_q(sl(n+m))$.
(b) If $q_i = q$ and $\pi_i = 0$ for $i = 1, \ldots, n$, $\pi_i = 1$ for $i = n+1, \ldots, n+m-1$, then again $\omega_{ij} = 1$ and the relations (17)–(26) will become the $U_q(sl(n|m))$ superalgebra in the Chevalley basis.
(c) If there is no restriction on the q_i's but $\pi_i = 0$ then the relations (17)–(27) become the algebra of $X_q(sl(n+m))$ which we have discussed in [1].
(d) With a special format for $q_i = q$ for $i = 1, \ldots, n$, $q_i = -q^{-1}$ for $i = n+1, \ldots, n+m-1$ and $\pi_i = 0$ for $i = 1, \ldots, n$, $\pi_i = 1$ for $i = n+1, \ldots n+m-1$ we obtain a superalgebra without any nilpotent elements.

In fact twisting and grading are very similar, by twisting or grading alone there exist nilpotent elements. But if twisting and grading occur simultaneously, nilpotency will be removed.

There is a well known relation between some of the solutions of QYBE and GYBE. Consider a particular solution of GYBE, $\hat{R}$ for which $\hat{R}_{ij,kl}$ is zero unless $\pi(\hat{R}_{ij,kl}) = \pi_i + \pi_j + \pi_k + \pi_l = 0$, then

$$R_{ij,kl} = (-1)^{\pi_i \pi_j} \hat{R}_{ij,kl} \tag{28}$$

solves the QYBE. It can be shown that if $\hat{R}$ is associated with $U_q(sl(n|m))$ then R is associated with $X_q(sl(n+m))$ [1] and if $\hat{R}$ is associated with $X_q(sl(n|m))$ then R is associated with $U_q(sl(n+m))$. We used the multiparametric non-standard R-matrix of $sl(n+m)$ [1] to obtain the multiparametric $\hat{R}$-matrix of $sl(n|m)$ and then constructed the multiparametric quantization of $sl(n|m)$ [6]. The same $\hat{R}$-matrix has also been obtained in [7]. In the same way, it is straightforward to use the multiparametric R-matrix of $sl(n+m)$ to construct the multiparametric version of $X_q(sl(n|m))$.

4. Universal R-matrix for $X_q(Gl(1|1))$

All the solutions of QYBE for the two-dimensional case were given in [8]. One of the solutions is the non-standard R-matrix of $GL(2)$ which is related to the $\hat{R}$-matrix of $GL(1|1)$ via relation (28). As an illustration of the relation between twisting and grading, we derive some quantum algebras corresponding to the following R-matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix}. \tag{29}$$

For the following choices of ζ, solutions of the QYBE and the GYBE, which are obtained lead to standard or non-standard quantum (super) groups.

If $\zeta = q$, $\pi_i = 0$ then R is a standard solution of QYBE, the associated QG is $U_q(Gl(2))$.

If $\zeta = -q^{-1}$, $\pi_i = 0$ then R is an exotic solution of QYBE, the associated QG is $X_q(Gl(2))$.

If $\zeta = q^{-1}$, $\pi_1 = 0$, $\pi_2 = 1$ then R is a standard solution of GYBE, the associated QG is $U_q(Gl(1|1))$.

If $\zeta = -q$, $\pi_1 = 0$, $\pi_2 = 1$ then R is an exotic solution of GYBE, the associated QG is $X_q(Gl(1|1))$.

It can be easily shown that the corresponding braid matrix B for the first and fourth case (and also for the second and third case) are the same and satisfies the following relation

$$B^2 = (q - q^{-1})B + 1.$$

The FRT *equations together with the following ansatz for L^+ and L^- matrices*

$$L^+ = \begin{pmatrix} q^{-H/2-K/2} & \sqrt{\zeta}\,\omega\zeta^{-H/2+K/2}X^+ \\ 0 & \zeta^{H/2-K/2} \end{pmatrix} \tag{30}$$

$$L^- = \begin{pmatrix} q^{H/2+K/2} & 0 \\ -\omega\sqrt{q}\,\zeta^{H/2-K/2}X^- & \zeta^{-H/2+K/2} \end{pmatrix} \qquad \omega \equiv q - q^{-1} \tag{31}$$

lead to the following quantum algebra

$$[H, X^\pm] = \pm 2X^\pm \qquad [K, \ldots] = 0 \qquad (\epsilon_1 q^{\epsilon_1} - \epsilon_2\zeta)(X^\pm)^2 = 0 \tag{32}$$

$$[X^+, X^-] = \frac{(q\zeta)^{H/2}(q/\zeta)^{K/2} - (q\zeta)^{-H/2}(q/\zeta)^{-K/2}}{q - q^{-1}} \tag{33}$$

where $[a, b] \equiv ab - (-1)^{\pi(a)\pi(b)}ba$. The comultiplications are

$$\Delta(H) = 1 \otimes H + H \otimes 1 \qquad \Delta K = K \otimes 1 + 1 \otimes K \tag{34}$$

$$\Delta(X^+) = X^+ \otimes 1 + (q\zeta)^{-H/2}(q/\zeta)^{-K/2} \otimes X^+ \tag{35}$$

$$\Delta(X^-) = 1 \otimes X^- + X^- \otimes (q\zeta)^{H/2}(q/\zeta)^{K/2}. \tag{36}$$

All the above algebras, which we denote by A in the following, are quasitriangular, that is there exist universal R-matrices, $R \in A \otimes A$ which intertwine Δ and $\Delta' \equiv \sigma_0\Delta$, and also have the following properties

$$(1 \otimes \Delta)R = R_{13}R_{12} \tag{37}$$

$$(\Delta \otimes 1)R = R_{13}R_{23}. \tag{38}$$

For the standard cases quasitriangularity has already been shown [9–11].

In the following we shall give the universal R-matrix for the algebra (32)–(36). For $\zeta = q$ or q^{-1} this R-matrix reduces to the R-matrix of $U_q(Gl(2))$ and $U_q(Gl(1|1))$ respectively, and for $\zeta = -q^{-1}$ and $-q$, it gives the universal R-matrix of $X_q(Gl(2))$ and $X_q(Gl(1|1))$. The R-matrix has the multiplicative form $R = \overline{R}\mathcal{K}$

$$\mathcal{K} = (q\zeta)^{1/4(H\otimes H+K\otimes K)}(q/\zeta)^{1/4(H\otimes K+K\otimes H)} \tag{39}$$

$$\overline{R} = \exp_p[-\omega(X^+\otimes X^-)] \qquad p = \begin{cases} -q\zeta & \text{for the non-graded case} \\ q\zeta & \text{for the graded case} \end{cases} \tag{40}$$

where $\exp_p$ is the p exponential function [11]

$$\exp_p(x) \equiv \sum_{n\geqslant 0} \frac{x^n}{(n)_p!} \qquad (n)_p! = (1)_p(2)_p\cdots(n)_p \qquad (n)_p = \frac{1-p^n}{1-p}. \tag{41}$$

As a check for the the validity of this R-matrix, we show that it actually intertwines ΔX^+ and $\sigma_0\Delta X^+$ in the algebra (32)–(36). By a straightforward calculation one can show that

$$\mathcal{K}(\Delta X^+)\mathcal{K}^{-1} = X^+\otimes (q/\zeta)^{K/2}(q\zeta)^{H/2} + 1\otimes X^+ \tag{42}$$

and

$$\begin{aligned}\overline{R}(X^+ &\otimes (q/\zeta)^{K/2}(q\zeta)^{H/2} + 1\otimes X^+)\\ &= (X^+\otimes (q/\zeta)^{-K/2}(q\zeta)^{-H/2} + 1\otimes X^+)\overline{R}\\ &= \sigma_0\Delta X^+\overline{R}\end{aligned} \tag{43}$$

so $R\Delta X^+R^{-1} = \overline{R}\mathcal{K}\Delta X^+\mathcal{K}^{-1}\overline{R}^{-1} = \Delta' X^+$. The same is true for other generators. By using

$$(\Delta\otimes 1)q^{H\otimes H} = q^{H\otimes 1\otimes H}q^{1\otimes H\otimes H} \tag{44}$$

one can verity that (37) and (38) are also satisfied.

In [4] it has been shown that $X_q(Gl(2))$ can be superized to $U_q(Gl(1|1))$. We think the same is true for $U_q(Gl(2))$ and $X_q(Gl(1|1))$ and also more generally for the case $X_q(sl(n+m))$ and $U_q(sl(n|m)$ $(U_q(sl(n+m))$ and $X_q(sl(n|m)))$.

Acknowledgments

The author wishes to thank V Karimipour, F Ardalan and S Rouhani for valuable discussions.

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